Solution of the Incompressible Laminar Boundary-Layer Equations

A. M. O. Smith* and Darwin W. Clutter†
Douglas Aircraft Company, Inc., Long Beach, Calif.

A general method is presented for calculation of the incompressible steady laminar boundary layer around either a two-dimensional or an axially symmetric body. The method is a modification of the Hartree-Womersley method. It consists of replacing the partial derivatives with respect to the flow direction by finite differences, while retaining the derivatives in a direction normal to the boundary, so that the partial differential equation becomes approximated by an ordinary differential equation. Reasons for choosing this method rather than the more conventional finite-difference methods are discussed. The method has been programed for an electronic computer, and solutions for a variety of flows are presented. Comparisons are made with other exact and approximate methods of solution. They include cases of flow separation, mass transfer, and discontinuities in the boundary conditions. A study of the response of the boundary layer to local boundary conditions is presented. Some comparison with experimental measurement is made also. The large number of calculations and comparisons establish that the method is rapid, accurate, and powerful. It appears capable of solving any flow problem for which the boundary-layer equations themselves remain valid. The only exception is a flow with a discontinuity in a boundary condition, where the method cannot produce an answer immediately downstream of the discontinuity but does appear to be precise a short distance farther downstream.

Nomenclature

f	= nondimensional stream function, defined by Eq. (5)
H	= boundary-layer shape parameter, δ^*/θ
m	= exponent in freestream velocity variation for similar flow, $U = cx^m$
M	= $(x/U)(dU/dx)$
n	= count of the number of steps in the x-wise direction
r	= radius of body of revolution
R	= $(x/r)(dr/dx)$
u	= x component of velocity in the boundary layer, $u/U = f'$
U	= velocity of the main stream at edge of boundary layer
v	= y component of velocity
x	= distance along surface measured from the stagnation point
y	= distance perpendicular to the wall
δ^*	= displacement thickness
Δ^*	= dimensionless displacement thickness, Eq. (11)
η	= transformed y coordinate in the boundary layer, Eq. (5)
θ	= momentum thickness
Θ	= dimensionless momentum thickness, Eq. (13)
μ	= dynamic viscosity
ν	= kinematic viscosity
ρ	= density
τ	= shear stress at wall
ψ	= stream function, Eqs. (1) and (2)

Subscripts

n	= evaluated at station n
w	= evaluated at wall
∞	= evaluated at undisturbed freestream, or a reference value
0	= reference value
sep	= evaluated at separation
primes	= differentiation with respect to η; $(\partial f/\partial\eta) = f'$, etc.

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* Supervisor, Aerodynamics Research Group, Aircraft Division. Associate Fellow Member AIAA.

† Design Specialist, Aerodynamics Research Group, Aircraft Division. Member AIAA.

Introduction

THERE are many problems of boundary-layer flow for which no satisfactory solutions have been found. Among them are questions of high-speed heat transfer, location of transition, and the effects of wall heat transfer and wall mass transfer on drag and separation. Development of a method to handle such problems is the ultimate objective of the work reported here. When it was started, the available methods for solving the boundary-layer equations appeared inadequate. Integral methods can treat general flows, but they give only approximate solutions. Similar flow methods are accurate but are restricted to special pressure distributions. Available finite-difference methods appeared to require long computing times for accuracy. Therefore, studies were made to develop a practical method for solving "exactly" the complete equations of compressible boundary-layer flow in two dimensions. The objective was to find a method capable of obtaining solutions for arbitrary values of 1) pressure distribution, 2) wall suction distribution, 3) gas properties, and 4) wall temperature distribution. The only restriction on the body is that it be two-dimensional or axially symmetric.

It appeared wise to approach the compressible problem by attacking the simpler incompressible flow first. Because no practical or well-developed method for solving even the incompressible equation exists, such a method has great importance for its own sake. A second purpose of the investigation was to learn as much as possible about the nature of the solutions, the effect of boundary conditions on the solution, and the relation of these solutions to true physical flows. The mathematics of the method of solution have been reported in Ref. 1; the application of the method to physical flows is reported in Ref. 2. This paper is a condensation of the latter. Since these studies were completed, the method has been applied to the compressible boundary layer, and preliminary studies indicate that the method is quite successful. However, this subject is to be left for a later report.

Boundary-Layer Equations

The equations to be solved cover the case of axisymmetric, steady flow about a body of revolution. The simpler prob-

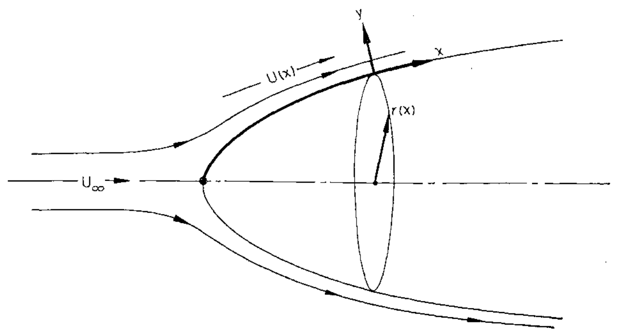


Fig. 1 Coordinate system. Boundary layer on a body of revolution.

lem of two-dimensional flow is included in the equations and is found merely by letting r, the body radius, be a constant. The basic notation and scheme of coordinates is shown in Fig. 1. The basic forms of Prandtl's boundary-layer equations for the case of steady axisymmetric flow can be obtained from any standard treatise, for example, Ref. 3. The continuity and momentum equations can be combined into a single third-order equation in terms of the stream function ψ by means of the relations

$$u = \frac{1}{r}\frac{\partial(\psi r)}{\partial y} = \frac{\partial \psi}{\partial y} \tag{1}$$

$$v = -\frac{1}{r}\frac{\partial(\psi r)}{\partial x} = -\frac{\partial\psi}{\partial x} - \frac{\psi}{r}\frac{dr}{dx} \tag{2}$$

The resulting equation is

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} - \left(\frac{\partial\psi}{\partial x} + \frac{\psi}{r}\frac{dr}{dx}\right)\frac{\partial^2\psi}{\partial y^2} = U\frac{dU}{dx} + \nu\frac{\partial^3\psi}{\partial y^3} \tag{3}$$

with the boundary conditions

$$\begin{aligned} y = 0&:\quad (\partial\psi/\partial y)_w = 0,\ [\partial(\psi r)/\partial x]_w = -rv_w \\ y \to \infty&:\quad \partial\psi/\partial y \to U(x) \end{aligned} \tag{4}$$

As it stands, Eq. (3) probably could be solved by the method to be described, but there are a number of difficulties. For instance, getting started is a problem because there is a singularity at $x = 0$, U is a variable quantity, and the boundary-layer thickness varies greatly with distance. A far more convenient form is obtained when Eq. (3) undergoes a transformation first introduced by Falkner and Skan.[4] Two new variables are introduced, a dimensionless height η and a dimensionless stream function f:

$$\eta = (U/\nu x)^{1/2} y \qquad \psi = (U\nu x)^{1/2} f(x,\eta) \tag{5}$$

If relations (5) are introduced into Eq. (3), and if $\partial f/\partial\eta$ is represented by f', etc., the following equation is obtained:

$$f''' = -\left(\frac{M+1}{2} + R\right)ff'' + M(f'^2 - 1) + x\left[f'\frac{\partial f'}{\partial x} - f''\frac{\partial f}{\partial x}\right] \tag{6}$$

The term $M = (x/U)(dU/dx)$ is a pressure gradient parameter, $R = (x/r)(dr/dx)$ is a similar measure of body radius, and f' turns out to be the velocity ratio u/U. The boundary conditions are

$$\eta = 0:\ f_w' = 0 \tag{7a}$$

$$f_w(x) = \frac{-(\nu U x)^{-1/2}}{r(x)}\int_0^x r(\xi)v_w(\xi)\,d\xi \tag{7b}$$

where ξ is a variable of integration. For an impermeable wall, $f_w = 0$. The third boundary condition is

$$\eta \to \infty: \qquad f' \to 1 \tag{7c}$$

A few of the more important properties of Eq. (6) are noted. If the edge velocity distribution is of the form $U = c\,x^M$, $(x/U)(dU/dx)$ reduces to M, the value of the exponent. Likewise if $r = k\,x^R$, $(x/r)(dr/dx) = R$. When $R = 0$, and M is a constant, it can be shown that the bracketed expression containing the partial derivatives with respect to x equals zero and that the following ordinary differential equation in η is obtained (Ref. 4, pp. 178, 179):

$$f''' = -[(M+1)/2]ff'' + M(f'^2 - 1) \tag{8}$$

The well-known families of "similar" solutions come from this equation.

Equation (6) has several advantages over other possible forms. Other methods usually have a singularity at $x = 0$ and require that an initial profile be specified, usually at some distance from the leading edge. Equation (6) has very good behavior at the origin. Here, provided that the derivatives of f are finite, the term in brackets which contains the x derivatives disappears and the equation reduces to that for a similar flow. Equation (6) is of such a form that the term in brackets is exhibited as a correction to the similar flow solution. The outer boundary condition is somewhat simpler, being always $f'(\infty) = 1$, whereas in x,y coordinates, it is $U(x)$.

Method of Solution

The fundamental idea for the solution, that of replacing the x derivatives by finite differences, in order to approximate the partial differential equation by an ordinary differential equation, was advanced by Hartree and Womersley.[5] The idea was applied to the incompressible boundary layer by Hartree,[6,7] Manohar,[8] and by the authors.[1,2,9] The x derivatives are only of first order in the equations being solved.

The authors[1] made an extensive study of the use of two-point, three-point, and four-point finite differences to represent the x derivatives in solving the same boundary-layer equations. The investigation showed that the three-point and four-point forms are both accurate and stable. The three-point form is used in the solutions presented here.

The basic scheme of the finite-difference representation is diagramed in Fig. 2. The two parameters R and M in the equation and the wall boundary condition f_w are arbitrary functions of x. As sketched, f_w may have a discontinuity in derivative whereas both M and R may have discontinuities in values. The space is divided into a number of regions bounded by lines x_n, x_{n-1}, x_{n-2}, etc. The spacing Δx need not be constant. Because the equation is parabolic in x, the problem must be solved by proceeding in the direction of positive x. It is assumed that the solution has been found at all previous stations up to and including x_{n-1}, which, of course, means that $f(\eta)$ and its derivatives are fully known

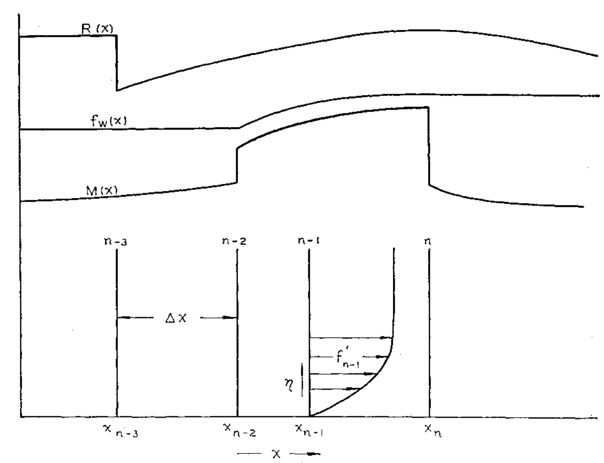


Fig. 2 Coordinate and notation system.

at these stations. The problem is to find the solution $f(\eta)$ at the new station x_n.

The two x derivatives in Eq. (6) are replaced by finite-difference Lagrangian formulas for three points as follows. Here, let m indicate the order of the derivative:

$$\left(\frac{\partial f^{(m)}}{\partial x}\right)_n = \left[\frac{1}{x_n - x_{n-1}} + \frac{1}{x_n - x_{n-2}}\right] f_n{}^{(m)} - \left[\frac{x_n - x_{n-2}}{(x_n - x_{n-1})(x_{n-1} - x_{n-2})}\right] f_{n-1}{}^{(m)} + \left[\frac{x_n - x_{n-1}}{(x_n - x_{n-2})(x_{n-1} - x_{n-2})}\right] f_{n-2}{}^{(m)} \quad (9)$$

which has an error like

$$[(\Delta x)^2/3](\partial^3 f^{(m)}/\partial x^3)$$

When these substitutions for the derivatives are made, it is assumed that all the other quantities in the boundary-layer equations are evaluated at x_n. The equation is then an ordinary differential equation in η with the variable quantities f and f' at the $n - 1$ and $n - 2$ stations. Step length Δx is not a primary parameter; instead, $x/\Delta x$ is. For further discussion of the errors associated with the finite-difference representation, see Ref. 1.

Method of Calculation

The solution of Eq. (6) is made difficult by both the nonlinearity and the fact that one boundary condition is at $\eta = \infty$. The existence of a solution and procedure for finding it are discussed in detail in Ref. 1. Briefly, the method of solution involves choosing values of f_w'' and integrating the equation directly, requiring that the value of f' remain within certain bounds. Once three such integrations have been completed all the way to some value called η_∞, which is an input in the program, a three-point interpolation procedure is used to determine the solutions that satisfy the outer boundary condition, i.e., $f' \to 1$ as $\eta \to \eta_\infty$. Integration was performed by a predictor-corrector type method that uses the Falkner multiple integration-extrapolation formulas and the Adams-type multiple integration-interpolation formulas.[10]

Reasons for Selection of Hartree-Womersley Method

The purpose of the present report is to describe a method of solving the boundary-layer equations and to prove its use by applying it to several incompressible flows. The purpose is not to present a treatise on existing and possible methods of solving the problem. Nevertheless, some answer should be given to the obvious question as to why the present method was chosen in preference to other ways of solving the equations.

The choice has two essential aspects. The first is the question of the form of the boundary-layer equation to solve. The second is the question of method to be used in solving the chosen equation. Consider the first aspect, the equation. Reasons for transforming the boundary-layer equation to x,η coordinates, and the resulting advantages were just discussed briefly. Perhaps the most useful of the advantages is the behavior of the equation at $x = 0$; no singularity occurs here as in other transformations, the boundary-layer thickness remains finite, and the x derivatives disappear, thus making the start of the solution extremely simple. Also the transformation removes most of the variation of boundary-layer thickness with x.

Various other transformations have been introduced by previous investigators to simplify the solution of the boundary-layer equations. For example, both Crocco's and Von Mises' transformations reduce Eq. (3) to a second-order equation. But, since one of the objectives of the present work was to develop a method of calculation which could be extended to compressible flow, the behavior of the transformations also had to be examined at high speeds. Each of the various transformations has difficulties. For example, that of Crocco uses the shear stress as one of the dependent variables, which can become double-valued in compressible flow. Von Mises' transformation leads to a singularity at the wall.

After examination of various methods for calculation of the boundary-layer equations, a modified Hartree-Womersley technique was chosen for the following reasons. The requirement that the method solve accurately any problem for which the boundary-layer equations are valid seems to call for a finite-difference technique, which is exact in the limit. In the finite-difference technique two procedures exist, the explicit and the implicit.

The explicit procedure has been applied to the boundary-layer problem by several investigators, for example, Baxter and Flügge-Lotz,[11] Raetz,[12] and Wu.[13] Explicit methods are inherently unstable. In order to obtain stable, accurate solutions, restrictions on the step sizes are necessary. In all of the preceding references, these restrictions appear to lead to excessive computation times for high accuracy.

Implicit finite-difference methods have no stability problem. When the present work was started, the only application of this method to the boundary-layer problem, known to the authors, was that of Kramer and Lieberstein.[14] They had used the Crocco form of the boundary-layer equation, whose disadvantages were just discussed. The method has more recently been applied by Flügge-Lotz and Blottner[15] to the compressible boundary layer. Their method appears to be both accurate and rapid. Conceptually, the Hartree-Womersley method is different from the classical implicit finite-difference procedures because it divides the region of the boundary layer into vertical strips, whereas the latter divides it into rectangles. However, since the resulting ordinary differential equation must be solved numerically, it can be said equally well that the Hartree-Womersley method is an ordinary implicit finite-difference method, set off by the fact that it has a special, unique method of finding the unknown at the next downstream station.

In addition to being inherently stable, the method chosen has the added advantage of reducing the equation to an ordinary differential equation. Questions of the existence of solution, of the nature of the solution, and of error propagation are much better understood for ordinary differential equations than they are for the complex of numbers arising from the more conventional finite-difference representation. A final reason for the choice of the Hartree-Womersley method is that it is known to produce results, has been moderately well explored, and produces answers of high accuracy.

Displacement and Momentum Thicknesses and Skin Friction

The conventional boundary-layer thicknesses are given by the well-known formulas; for displacement thickness,

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy \quad (10)$$

or, in nondimensional form,

$$\Delta^* = \frac{\delta^*}{x}\left(\frac{Ux}{\nu}\right)^{1/2} = \int_0^\infty (1 - f')d\eta \quad (11)$$

For momentum thickness,

$$\theta = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dy \quad (12)$$

or, in nondimensional form,

$$\Theta = \frac{\theta}{x}\left(\frac{Ux}{\nu}\right)^{1/2} = \int_0^\infty f'(1 - f')d\eta \tag{13}$$

The local skin friction, defined as the ratio of local shear stress at the wall to the local dynamic pressure outside the boundary layer, is given by

$$c_f = \frac{\tau}{q} = \frac{\mu(du/dy)_w}{\frac{1}{2}\rho U^2} = 2\left(\frac{\nu}{Ux}\right)^{1/2} f_w'' \tag{14}$$

Accuracy of the Method of Solution

A large number of flows have been calculated in order to establish the accuracy of the method of solution.[1,2] Accuracy primarily depends on two aspects of the method, integration in the η direction and the step size in the x direction.

Accuracy of Integration in the η Direction

Since the most exact solutions to the boundary-layer equations are for similar flows, such flows were first used to check the accuracy of the present method in solving the equation in the η direction. (Solutions for such flows accurate to five and six decimal places are published.[16]) It should be noted that checking the method of solution for similar flows does not necessarily prove it for nonsimilar flow. In studying the literature, the authors found that often a method developed for calculating the general boundary-layer equation is tested by calculating a similar flow; and, then, because these simple calculations work, the investigator claims the method is proved. From the authors' investigation of several methods of calculation, they have found that methods that produce accurate solutions for similar flows may fail to give answers for nonsimilar flows.[1] But, similar flows do provide an accurate check on that portion of the present method that involves solution of the equation in the η direction.

The present method duplicated the known similar solutions to any specified accuracy, including the separation profile and flows with mass transfer.[2] This investigation of similar flow showed that the maximum error in all calculated values of f, f', etc., is in the value of f_w''. For this reason and because it is a measure of the skin friction, f_w'' is the principal parameter used to present and compare solutions that follow.

Effect of Step Size in x on Solution Accuracy

Since exact solutions for nonsimilar flows are practically nonexistent, the accuracy of the present method for such flows was investigated by studying convergence of the solutions as Δx was varied. In Eq. (6) the x derivatives of f and f' appear only in the last term:

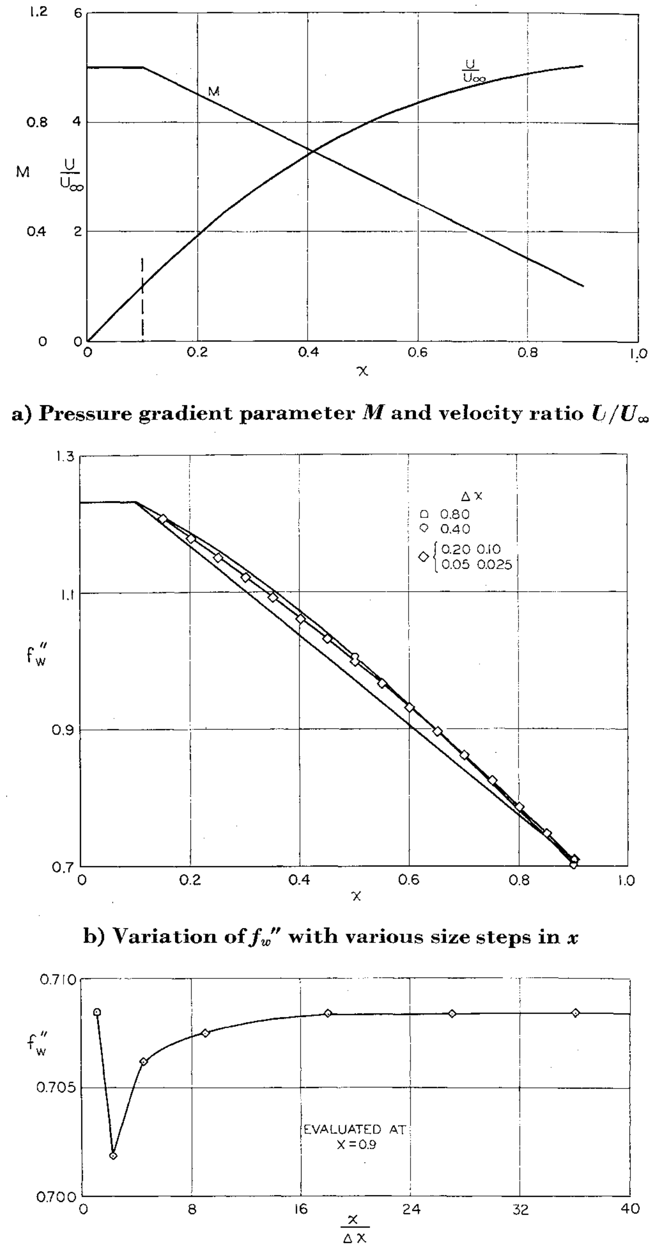
$$x[f'(\partial f'/\partial x) - f''(\partial f/\partial x)]$$

In the finite-difference scheme, this term is essentially replaced by a term of the form

$$(x/\Delta x)[f'\Delta f' - f''\Delta f]$$

Now as Δx approaches zero, $\Delta f'$ and Δf would approach zero in an exact solution, but in the computer program round-off errors exist and, furthermore, these errors are multiplied by the quantity $x/\Delta x$. Therefore as Δx approaches zero, the error in the solution is not necessarily decreased.

The effect of variation of Δx on the solution was studied for a variety of flows which were selected to include high rates of change in the pressure-gradient parameter M, both adverse and favorable gradients, suction and blowing at the surface, and discontinuities in the boundary conditions of M and v_w. Of course, it would be physically impossible to have discontinuities in the boundary conditions, and furthermore the assumptions that reduce the Navier-Stokes equations to the boundary-layer equations are not valid with such discontinuities. The purpose of studying such extreme conditions was to determine further what step sizes in Δx were required for convergence. A criterion of convergence established for these cases should be conservative for ordinary physical flows. Results of one of the cases studied are presented in Fig. 3. The effect of Δx on the convergence of solution that is shown is typical of all cases studied. Calculated values of f_w'' vs x are shown for various step lengths, and values of f_w'' at a given x station as calculated for various Δx's are shown. At first, as Δx is decreased, the values of f_w'' appear to converge, but, if $x/\Delta x$ becomes too large, the values diverge. As was anticipated in the previous paragraph, the divergence is brought about because the quantity $x/\Delta x$ multiplies the round-off errors. In the cases studied, divergence did not occur for values of $x/\Delta x$ less than about 25. The maximum error that has been observed when divergence occurs is one in the fifth significant digit for $x/\Delta x \simeq 25$. When $x/\Delta x$ becomes very large, the values of f, f',



a) Pressure gradient parameter M and velocity ratio U/U_∞

b) Variation of f_w'' with various size steps in x

c) Effect of step size in x on f_w'' at $x = 0.9$

Fig. 3 Effect of step size in x on solution for a flow with gradual decrease in M following a short length of stagnation-type flow. $M = 1.0, x \leqq 0.1$; $M = 1.1 - x, x \geqq 0.1$.

Table 1 Calculated values of f_w'' for Howarth's retarded flow, $U = 1 - \frac{1}{8}x$

x	Present method f_w''	Hartree's solution
0	0.332057	...
0.2	0.29091	...
0.4	0.24392	0.2440*6*
0.8	0.1168*6*[a]	0.116*8*
0.88	0.0777*7*	0.077*3*
0.92	0.0520*6*	0.050*8*
0.948	0.02*640*	0.02*49*
0.956	0.01*427*	0.01*14*
0.958	0.009*53*	0.00*59*
0.9589	0.00*647*	0-extrapolated
0.960	0-extrapolated	

[a] Italic digits may not be significant. Note, not all steps are shown for $x < 0.948$.

etc., at the outer edge of the boundary layer are very sensitive to the values of f'' at the wall. In some cases a change of one in the seventh decimal in f_w'' can cause a change in the second decimal place of f' near the outer edge of the boundary layer. The sensitivity can be related to the magnitude of $x/\Delta x$ and can be explained partially by the asymptotic nature of the boundary layer at large η.[1]

In all the examples studied it was found that values of f_w'' could be calculated accurately to at least five decimal places, providing the quantity $x/\Delta x$ was less than 25. This is not a serious restriction, since it permits the boundary layer to be determined at about 25 equal-length stations in x. Since the boundary-layer thickness grows parabolically with x, the step length in x can usually grow parabolically with no loss in accuracy, and thus the restriction is even less serious. Also the restriction can be lessened by using "double-precision" routines in the computer program. In the region of convergence, it was found that the error in f_w'' is about quadratic in Δx, that is,

$$\Delta f_w'' = |f_w'' - f''_{w_{\text{exact}}}| \sim (\Delta x)^2$$

This could have been expected, since the error in replacing the x derivatives by the three-point finite-difference form results in a quadratic error. The same variation of the solution with Δx was found for the flows with discontinuities in the boundary conditions. Examples of discontinuities are given in following sections.

Behavior of Solution near Separation

The behavior of the method of solution near a separation point was studied for the velocity field given by

$$U = 1 - \tfrac{1}{8}x$$

This flow has been studied by Howarth,[17] von Karman and Millikan,[18] Hartree,[6] Leigh,[19] and many others. Both Hartree and Leigh used a method similar to that presented here. Leigh extended the work of Hartree by using an electronic computer to make a more thorough study of the solution in the region of separation. As a test case, the flow thus has the advantages both of having been studied extensively and of leading to separation.

The values of f_w'' calculated by the present method are compared with those of Hartree in Table 1. The step sizes in x were chosen to give convergence of f_w'' to at least three decimal places at the last station calculated. The solution is believed to be more accurate at the forward station, as indicated in Table 1. Hartree presents his values as accurate to four decimal places, but this is questionable. He carried only four decimal places throughout his calculation. In the present investigation it has been found that small errors at the upstream stations can be greatly magnified in downstream stations. For example, an error of one in the sixth decimal place of f_w'' at the initial station can produce a difference of about one in the second decimal place near separation, i.e., at $x = 0.958$. The same growth in error with x would cause erroneous results in Leigh's calculation of the same problem. For further details of this error growth, see Ref. 2.

The exact separation point, that is, the point where $f_w'' = 0$, cannot be calculated for the linearly retarded flow, because the boundary layer becomes singular there. This singularity has been discussed by Prandtl[20] and Goldstein.[21] Furthermore, Prandtl points out that the pressure distribution around the separation point cannot be taken arbitrarily but must satisfy certain conditions connected with the existence of backflow downstream of separation. That is, in a physical flow the external pressure gradient is always such that there is no singularity in the flow. Prandtl states that a necessary condition for separation without a singularity is $d^2U/dx^2 > 0$. Obviously the flow in question violates this condition. In the region of separation the solution becomes very sensitive to the value of f_w'', which makes it difficult to find the exact value that satisfies the outer boundary conditions. Behavior of the solution here is very similar to that which occurs when $x/\Delta x$ becomes large. This same sensitivity was found by Hartree and Leigh in their solution of the same problem.

Even though the separation point cannot be calculated, the point may be approached by taking small steps in x. But in the present method there is a limitation on the smallest step that can be taken, because of the divergence that occurs when $x/\Delta x$ is greater than about 25. The step sizes that were used in the calculation near separation are indicated in Table 1. The separation point obtained by extrapolation is $x = 0.960$. Hartree's value, also found by extrapolation, is $x = 0.9589$. Because of the limitation on the smallest step size that the method of calculation can handle accurately, the separation point was approached in a second way. Since application of suction at the wall is known to delay separation, the retarded flow was studied by applying various amounts of suction in the region of separation. The investigation showed that as the amount of suction, f_w, approached zero the separation point approached the location just given, $x = 0.960$.

In all the flows leading to separation which have been investigated by the present method, the behavior of the calculation near separation was similar to that just mentioned. Though the exact separation point could not be calculated, the point could always be determined by extrapolation within a half percent of x. In some of the flows $d^2U/dx^2 > 0$, and therefore, there would not necessarily be a singularity at separation according to the forementioned criterion. But even for these flows, the solution near separation behaved like that found for Howarth's retarded flow.

Comparison of Calculated Profiles with Experiment

One of the most accurate and complete sets of experimental data on incompressible laminar boundary-layer profiles on a body other than the flat plate is that measured by Schubauer on an elliptic cylinder. The boundary layer on this body was calculated by the present method with the pressure distribution observed by Schubauer. The numerical values of pressures which were used were obtained from Hartree,[7] who also calculated the boundary layer for this flow. Comparison of the calculated profiles with the measured ones are shown in Fig. 4a at four stations. Agreement is within the experimental accuracy of the measurements. Figure 4b shows the values of f_w'' calculated both by the present method and by Hartree. Up to $x = 1.4$, the values agree within three decimal places. Aft of this point, the values of Hartree fall below those of the present method. Hartree carried four decimal places in his calculation. In the present method the values of f_w'' were required to converge within six decimal places at each station. In the figures, all calculated values of f_w'' are shown, and therefore they indicate the step length used.

Neither of the calculations gives separation of the boundary layer, although Schubauer observed separation at $x = 1.99 \pm 0.02$. Hartree shows that a slight modification in the observed pressure distribution would produce separation in the observed region; the same is true of the present method. The modification is probably within the experimental accuracy of the pressure distribution observed by Schubauer. It is also possible that, because the boundary layer was so thick in the region of separation, the pressure change across the boundary layer is not negligible as was assumed in the development of Prandtl's boundary-layer equations.

Because the flow is typical of ones for which the method would be used in engineering practice, calculating times are presented. About 2.5 min of IBM 7094 machine time are required for the run. Machine time averages out to be about 0.08 min per station.

Calculation of Boundary Layer on Bodies of Revolution

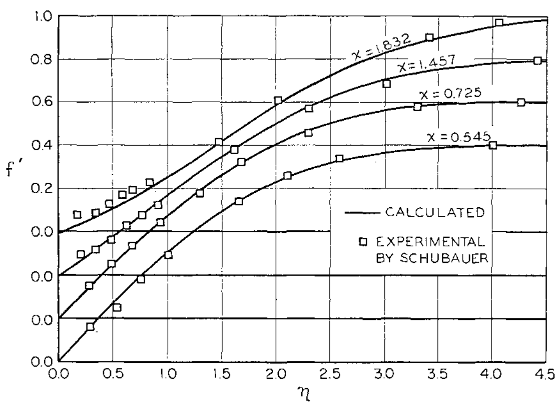
For axially symmetric bodies the term $R = (x/r)(dr/dx)$ appears in the momentum equation, but it adds no complication to the method of solution. As an example, the boundary layer over a sphere has been calculated. The velocity given by potential flow is used, x, as usual, being distance measured along the surface:

$$U/U_\infty = \tfrac{3}{2}\sin(x/R_1) = \tfrac{3}{2}\sin\theta$$

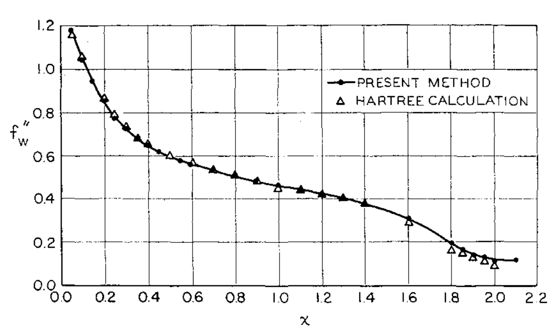
where R_1 is the radius of the sphere. Calculated values of f_w'' are given in Table 2 by the "axially symmetric" column.

The boundary layer over an axially symmetric body can be calculated by means of just the two-dimensional boundary-layer equations by use of Mangler's transformation. As a check on the accuracy of the present method and its computer program, the flow over the sphere was calculated a second time by using this transformation. The two calculations are compared in Table 2. Values agree to three decimal places or better forward of 60° and then diverge slightly. Separation, corresponding to $f_w'' = 0$, is found by extrapolation to be 105.7° and 105.9°, respectively, for the transformed flow and the true three-dimensional flow.

The sphere flow has also been calculated by a series method of calculation and is presented on pp. 187–190 of Ref. 3. There separation is found to be at 109.6°. The series method and its inherent errors are discussed in the following section.



a) Comparison of calculated and measured velocity profiles



b) Comparison of values of f_w'' calculated by the present method with those of Hartree

Fig. 4 Boundary layer on the elliptic cylinder studied experimentally by Schubauer.

Table 2 Comparison of values of f_w'' on a sphere as calculated by the axially symmetric equation and by Mangler's transformed equation

	f_w''	
$\theta°$	Axially symmetric	Transformed
0	1.31189	1.31188
30.0	1.25888	1.25886
60.0	1.08011	1.07970
90.0	0.6562	0.6422
100.0	0.3580	0.3349
105.7	...	0
105.9	0	...

Comparison of Present Method with Series Methods of Calculation

In 1935 Howarth developed a general solution for the velocity profile of the boundary layer subject to a mainstream velocity, expressible as a power series in terms of the distance from the stagnation point. He calculated and published the universal functions required for solution for an odd power series up to the x^7 term. Tifford expanded the series to include the x^{11} term.[22]

The boundary layer over a circular cylinder was calculated by the series and the present method. The velocity distribution used was that observed by Hiemenz, who found that the distribution could be accurately represented by three terms in an odd power series (p. 150, Ref. 4),

$$U = A\theta - B\theta^3 - C\theta^5$$

where $A = 3.1657 \times 10^{-2}$, $B = 1.4383 \times 10^{-6}$, $C = 7.6247 \times 10^{-11}$, and θ is measured in degrees. Values of f_w'' are shown in Fig. 5. The step sizes in both η and x were selected so as to give f_w'' accurate to four decimal places in the present method of calculation. Forward of 50°, the values agree with Tifford's values to four decimal places, but near separation they do not even agree in the first decimal place. It can be shown that near separation the Tifford series is divergent, and probably truncation error causes the disagreement. Separation is indicated at 80.0° by the present method and at 83.0° by the series method. Experiments that have been carried out at sufficiently low Reynolds numbers to insure that the flow in the boundary layer up to separation is steady and not turbulent show separation at about 80°. At slightly higher Reynolds numbers the flow near separation becomes unsteady but supposedly remains laminar, and separation is observed at 82° or 83°.

Comparison of Present Method with Exact Solution for Nonsimilar Flow with Suction

If uniform suction is applied to a flat plate at zero incidence, the resulting profile asymptotically approaches a profile of constant thickness. The profile, known as the "asymptotic suction profile," is given by a simple exponential.[23] This flow was calculated by the present method, and the results agreed with the exact solution of Iglish[23] within the accuracy with which he presented his data, three decimal places.[2]

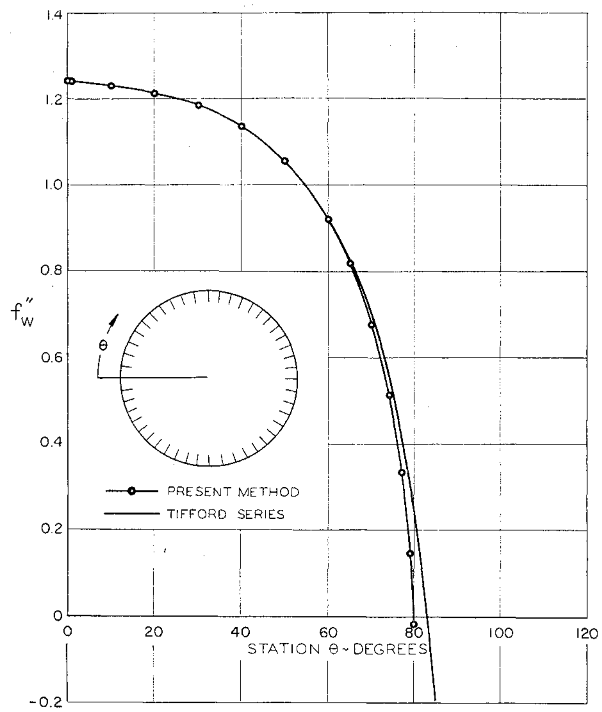
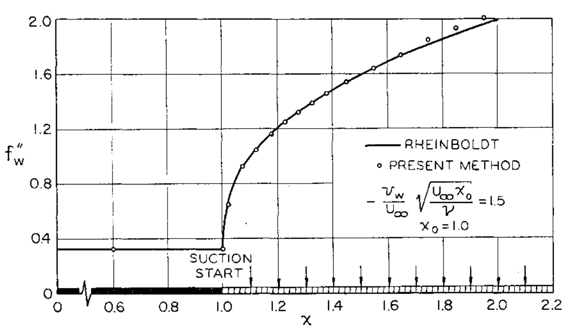
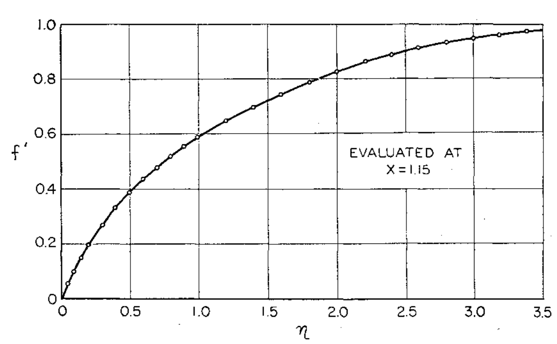


Fig. 5 Comparison between values of f_w'' on a circular cylinder as calculated by the present method and by the Tifford series method. Velocity distribution observed experimentally by Hiemenz.



a) Variation of f_w'' with distance

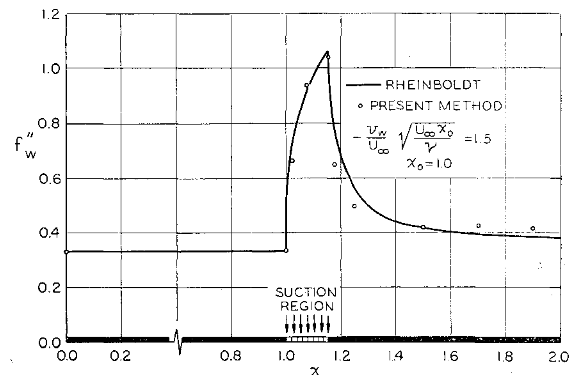


b) Velocity profile in region of suction

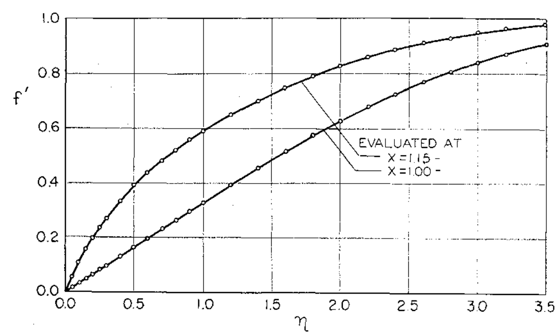
Fig. 6 Comparison of the present method with that of Rheinboldt for a discontinuity in suction on a flat plate. $f_w = 0$, $x \leqq x_0$; $f_w = -(v_w/U_\infty)(U_\infty x/\nu)^{1/2}[(x - x_0)/x]$, $x \geqq x_0$.

Besides the asymptotic suction flow, another accurate solution for nonsimilar flows is the work of Rheinboldt,[24] who calculated the boundary layer over a flat plate with discontinuous suction. Constant suction velocity is applied aft of a point $x = x_0$. Forward of x_0 the surface is impermeable. One of Rheinboldt's solutions is compared with that of the present method in Fig. 6.

Rheinboldt also calculated the boundary layer for a flow with impulse suction, that is, suction applied only over a short distance Δx. The flow is described by $U = U_\infty =$ const, $v_w = 0$ at $x < 1.0$ and $x > 1.15$, $v_w =$ const, and nonzero at $1.0 \leqq x \leqq 1.15$. The variations of f_w'' with distance



a) Variation of f_w'' with distance



b) Velocity profiles at an infinitesimal distance upstream of the start of suction and at an infinitesimal distance upstream of the end of suction

Fig. 7 Comparison of the present method with that of Rheinboldt for impulse suction on a flat plate. $v_w = 0$ for $x \leqq 1.0$ and $x \geqq 1.15$; $f_w = -(v_w/U)(Ux/\nu)^{1/2}[(x - x_0)/x]$ for $1.0 \leqq x \leqq 1.15$, $x_0 = 1.0$.

and velocity profiles at the start and at the aft end of the suction region are shown in Fig. 7. Just aft of the suction region, the values of f_w'' calculated by the present method differ in the second decimal place from those of Rheinboldt. It is believed that the former are in error. The error is introduced by the very short steps in x, which are necessary at the discontinuity in order to get the solution to converge farther downstream. The velocity profile calculated at the aft end of the suction region agrees with the results of Rheinboldt within the accuracy with which his curves could be read.

Effect of Previous History of the Boundary Layer on Separation

The effect of previous history of the boundary layer on separation has been studied by the present method. Five flows, shown in Fig. 8, were selected so as to give greatly different initial profiles. These initial flows are similar,

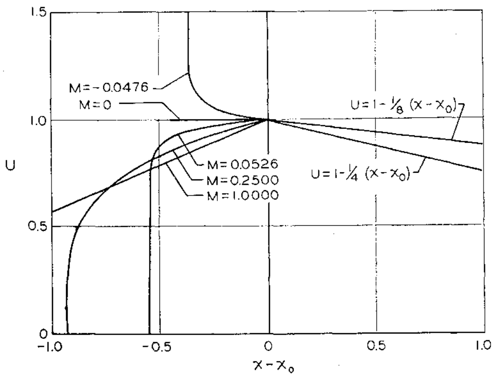
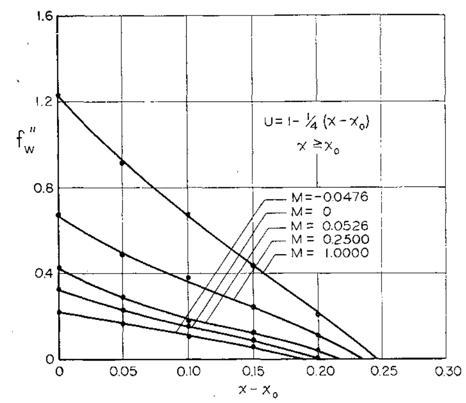
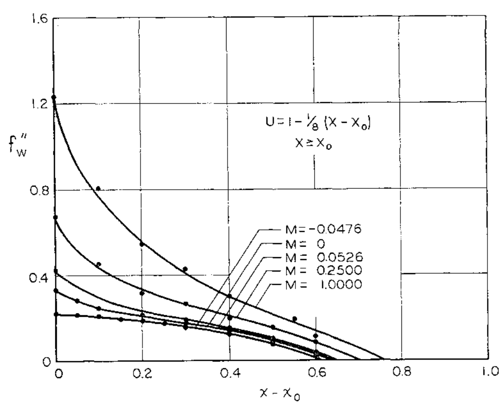


Fig. 8 Flows used to study effect of history of the boundary layer on its separation. Flows similar, forward of x_0; x_0 such that the momentum thickness at that point is the same for all flows.



a)

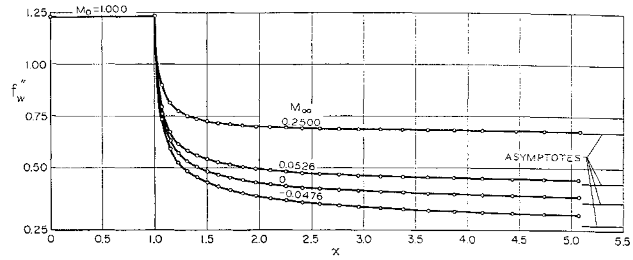


b)

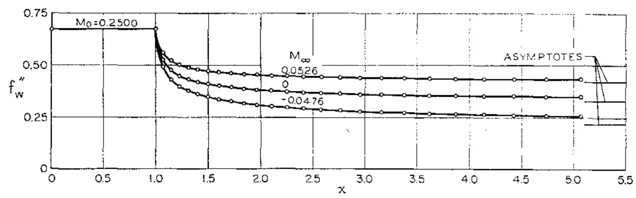
Fig. 9 Variation of f_w'' with distance, showing effect of initial flow on separation for two retarded flows: a) $U = 1 - \frac{1}{4}(x - x_0)$; b) $U = 1 - \frac{1}{8}(x - x_0)$.

M = 1.0, 0.25, 0.0526, 0.0, and −0.0476. These particular values were chosen because accurate solutions for them are available.[16] They correspond to Hartree's β of 1.0, 0.4, 0.1, 0.0, and −0.1, respectively. They also cover the range of M's encountered in most physical flows. The length of initial flow x_0 is chosen so that each of the flows develops the same momentum thickness θ at x_0. That is, $\theta^2 U/\nu = 0.2$, and the unit Reynolds number U/ν is the same. The velocity distribution aft of x_0 is identical for each of the flows and has the form

$$U = 1 - m(x - x_0) \qquad \text{for } x \geqq x_0$$



a)



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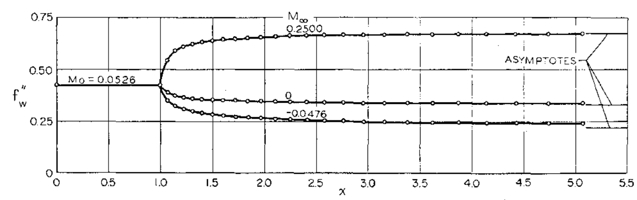
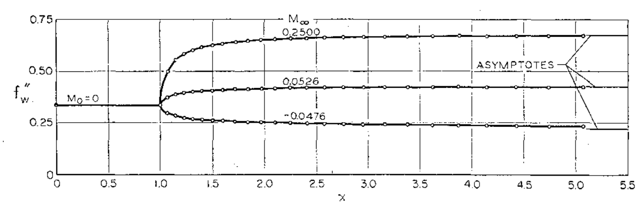
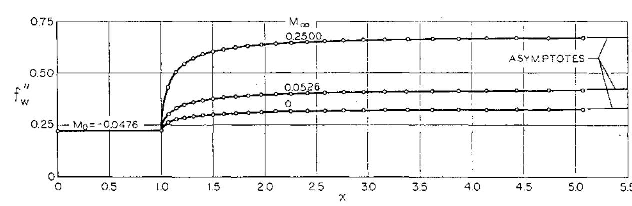


c)



d)



e)

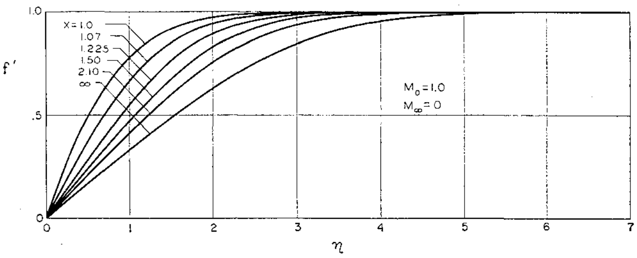
Fig. 10 Response of the shear parameter to local boundary conditions for the following types of flows: M = const = M_0 for $x \leqq 1.0$; M = const = M_∞ for $x > 1.0$.

The resulting variation of f_w'' with $(x - x_0)$ is shown in Fig. 9 for two values of m. Separation points for the various flows are indicated in the figure. For the flows shown in either Figs. 9a or 9b, the approximate methods for calculating laminar boundary layers that are one-parameter methods would predict the same separation point, irrespective of the different-shape profiles upstream. For example, Thwaites' method[3] predicts separation at x = 0.276 for all the flows shown in Fig. 9a; and x = 0.856 for those in Fig. 9b.

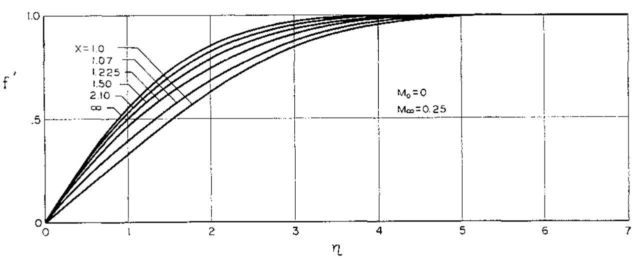
Table 3 Comparison of piecewise method with Hartree's and the present methods of solution, Schubauer's ellipse

	f_w''			$(\theta/c)(R)_c^{1/2}$		
x/c	Hartree	Present method	Piecewise	Hartree	Present method	Piecewise
0.2	0.8691[a]	0.86816[a]	0.8547	0.157	0.1596	0.168
0.6	0.5666	0.56639	0.5697	0.331	0.3312	0.332
1.0	0.4556	0.45572	0.4447	0.468	0.4665	0.470
1.2	0.4181	0.41868	0.4104	0.531	0.5295	0.532
1.4	0.3789	0.38086	0.3587	0.592	0.5895	0.595
1.6	0.2985	0.30123	0.2578	0.664	0.6594	0.658
1.8	0.1862	0.18945	0.0695	0.748	0.7395	0.738
1.85	0.1539	0.15742	0	0.775	0.7660	0.758
1.95	0.1107	0.11991	Separation	0.825	0.7943	...

[a] Last digit presented in results of both Hartree's and the present method may not be significant.



a)



b)

Fig. 11 Response of velocity profile to local boundary conditions for two flows described in Fig. 10: a) $M_0 = 1.0$, $M_\infty = 0$; b) $M_0 = 0.0$, $M_\infty = 0.25$.

Adjustment of Boundary Layer to Local Boundary Conditions

The study on the effect of previous history on separation is closely connected to the question of how rapidly the boundary layer adjusts to local changes in boundary conditions. The answer to this question is of great importance in application of local-similarity methods that give approximate solutions to the boundary-layer equations. In local-similarity methods, solutions are obtained by a step-by-step procedure, in which each x-wise segment of the flow is approximated by one of the family of similar flows. The method assumes that the local boundary-layer shape and shear parameter are functions of the local flow only. For the method to be accurate, the boundary layer must adjust to the local boundary conditions very rapidly.

This adjustment has been studied by using the present method to observe the boundary-layer solution downstream of a change in pressure gradient. Flows studied begin with one of the similar profiles; then at some point downstream, $x = 1.0$, the velocity gradient abruptly changes. But there are no discontinuities in external velocity or other boundary conditions. These are the same assumptions used in the local similarity methods. Variation of the shear parameter f_w'' with distance downstream of the pressure change is presented in Fig. 10. Asymptotic values of f_w'' are also shown. The shear at first approaches the asymptotic value very rapidly, then slows, and gradually converges to the value. The variation is similar for all flows studied. Also the variation of the momentum thickness and shape parameter with distance is similar. Details of the adjustment of the profile to the local pressure change can be seen in Fig. 11, which shows velocity profiles at various downstream distances for two of the flows studied. Asymptotic profiles are also shown. The profiles adjust most rapidly near the wall and slowly approach the limiting values at the outer edge of the boundary layer. In conclusion, the study shows surprisingly slow adjustment of the boundary layer to local conditions.

Even though the response of the boundary layer to local boundary conditions is slow, local-similarity (sometimes called piecewise) methods may give adequate accuracy for engineering purposes for particular flows. As an example, the flow about Schubauer's ellipse which is discussed in the foregoing and presented in Fig. 4 was calculated by the piecewise method of Ref. 25. Results are compared with Hartree's solution and the present method in Table 3. Eighteen steps were used in the piecewise solution. It would appear that the piecewise method is quite adequate for many engineering uses, particularly in favorable pressure gradients. It is known to be in error near separation.[25]

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Effects of Exhaust Nozzle Recombination on Hypersonic Ramjet Performance: I. Experimental Measurements

Erwin A. Lezberg* and Leo C. Franciscus†
NASA Lewis Research Center, Cleveland, Ohio

Experimental temperature measurements have been made, in supersonic nozzles with cone exit half-angles of 10.5° and 7°, for hydrogen-air and methane-air combustion products at stagnation temperatures up to 5400°R and pressures up to 4.5 atm. The results indicate freezing in the vicinity of the nozzle throat. Comparisons of the data with an approximate freezing-point analysis and an exact calculation showed good agreement with both. Preliminary spectral absorption measurements for the OH $^2\Sigma^+$ -$^2\Pi$ electronic transition were made by using the curve of growth to determine concentration and to demonstrate feasibility of spectroscopic concentration measurements.

Nomenclature

A = cross-sectional area of nozzle
A^* = critical flow area
A_K = line strength
a = ratio of collision broadening to Doppler broadening, $[(b_N + b_c)/b_D](\ln 2)^{1/2}$
b_c = collision half-width, cm^{-1}
b_D = Doppler half-width of spectral line, cm^{-1}
b_N = natural half-width of spectral line, cm^{-1}
c = speed of light, cm/sec
F = $f(2J + 1)/A_K$
f = ratio of number of dispersion electrons to number of absorbers
h = Planck constant
J = rotational quantum number
k = Boltzman constant
l = length, cm
N = concentration, cm^{-3}
P_ω = absorption coefficient per molecule, cm^{-2}
p = static pressure
Q_r, Q_v = partition functions for rotation and vibration
R = radial distance
T = static temperature
W = equivalent width, cm^{-1}
X = optical depth Nl, cm^{-2}
δ^* = displacement thickness
φ = equivalence ratio
ω = wave number, cm^{-1}

Subscripts

A = air
eq = equilibrium
OH = hydroxyl radical
K = rotational level
u = upper electronic state
0 = stagnation state

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* Aeronautical Research Scientist.
† Aerospace Engineer, Mission Analysis Branch.

Introduction

PERFORMANCE of hypersonic ramjets using both subsonic and supersonic combustion can be dependent throughout parts of the flight corridor on kinetic rate processes in the exhaust nozzle.